A COMMENTARY ON SYMBOLIC INTEGRATION IN SOFTWARE

1. Introduction. Over 2000 years ago, Archimedes found formulas for the surface areas and volumes of solids such as the sphere and the cone. His method of " integration" was based on sums of powers of consecutive integers. Leibniz and Newton independently discovered calculus, they found that differentiation and integration are inverse operations, while Riemann and Lebesgue gave integration a firm bases. Liouville created in the 1830s a framework for "constructive" integration by finding out when indefinite integrals of elementary functions are again elementary functions.
In the late 1940s Joseph Fels Ritt extended the framework in his book "Integration in Finite Terms". Between 1969 and 1976 Robert Risch made a breakthrough by using algebraic methods in algorithmic integration with his work on the general theory of integrating elementary functions. His algorithm or rather his "decision procedure" does not apply to all classes of elementary functions. In the 1980s more progress was made in extending his method to certain classes of special functions, especially thanks to work by Moses, Davenport Geddes, Trager, Bronstein and others. This gave rise to software packages for symbolic integration in software like Macsyma, Reduce, Maple, Axiom, Mathematica, MATLAB and others. Most software packages have however some deficiencies in their integration routines, one example being the rather simple integrals of form

$$\int \frac{x}{\sqrt{(x^2+a)^2 - c(x+b)^2}} dx$$

with coefficients a, b and c satisfying $0 < \frac{a^2}{b^2} < c < 2a$ with $b \neq 0$. These integrals are elementary, while most software either do not solve them or give them wrongly as elliptic integrals. Let us study this particular case further.

2. A procedure. When integrating functions of the type

$$f(x) = \frac{P_m(x)}{\sqrt{P_n(x)}} , \qquad (1)$$

where $P_m(x)$ and $P_n(x)$ are polynomials in the variable *x* of degree m and n respectively and $n > 2m$, the general idea of Liouville´s [Liouville, 1833] applies. Here we are mainly interested in the case $n = 4$.
The above integrals have the form

$$F(x) = c \ln[g(x)\sqrt{P_n(x)} + h(x)] , \qquad (2)$$

where *g(x)* and *h(x)* are very simple polynomials of *x* and c is a constant. Instead of trying to perform the integration directly we proceed in an "algorithmic" way with the form given in equation (2). Then the derivative of *F(x)* is

$$F'(x) = c \; \frac{g'(x)\, P_n(x) + ½ g(x)\, P'_n(x) + h'(x)\sqrt{P_n(x)}}{\sqrt{P_n(x)}\,[\, g(x)\sqrt{P_n(x)} + h(x)\,]} \qquad (3)$$

and we get the integral in explicit form if we can solve for *g(x)* and *h(x)* satisfying the differential equations

$$c\, h'(x) = P_m(x)\, g(x) \quad \text{and} \quad c[g'(x)\, P_n(x) + ½ g(x)\, P_n'(x)] = P_m(x)\, h(x). \qquad (4)$$

As $P_n(x)$ and $P_m(x)$ are given polynomials of x, for example $P_n(x) = \sum_{i=0}^{n} \gamma_i x^i$, we now set $g(x) = \sum_i \alpha_i x^i$ and $h(x) = \sum_i \beta_i x^i$ to get recursive algebraic relations between the coefficients $\{\alpha_i\}$, $\{\beta_i\}$ and $\{\gamma_i\}$. For example the equations 4 read in terms of coefficient relations (in this particular case with $P_m(x) = x$)

$$(i+2)\, \beta_{i+2} = \frac{1}{c}\, \alpha_i \quad \text{and} \quad ½ \sum (i + k + 2)\, \alpha_i\, \gamma_{k-i+2} = \frac{1}{c}\, \beta_k \qquad (5)$$

These recursive relations define the integral function $F(x)$ and it should be rather simple to perform this process symbolically in computer software. If the recursive relations of equations 5 give no solution, then the simple form of $F(x)$ used in equation 2 is not the right one, $F(x)$ should then be of the more general Liouville-form [Ritt, 1948], given in equation 6

$$F(x) = v_0(x) + \sum c_i \, ln[v_i(x)] \ , \tag{6}$$

otherwise the integral is non-elementary [Ritt, 1948]. For example, equations 4 and 5 have no solution for the case {m=0, n=3}, and it should not have because that case leads to first type elliptic functions. Note that equations 4, two well-behaved coupled first order differential equations, in equation 5 become two recursive algebraic equations, having a unique solution when the integral is elementary and no solution when it is non-elementary. This is not proved here, but it follows from the theorems of Liouville, Ritt and Risch .
A simple example of this recursive procedure is given below.

3. Example. Find the anti derivative of the following function

$$f(x) = \frac{x}{\sqrt{(x^2+a)^2 - c\,(x+b)^2}} = \frac{x}{\sqrt{P(x;a,b,c)}} \tag{7}$$

The denominator is written in the given form to easily yield the poles of f(x). We have only three free parameters a, b and c since the cubic term is missing. Let a, b and c be rational numbers satisfying the condition (this condition may be too strong)

$$0 < \frac{a^2}{b^2} < c < 2a \ , \ (b \neq 0) \tag{8}$$

The anti derivative can be calculated in its simplest form as

$$F(x) = \int \frac{x}{\sqrt{P(x;a,b,c)}} \, dx = -\text{ constant} * ln\,[\, g(x)\sqrt{P(x;a,b,c)} - h(x) \,] + C \tag{9}$$

Forming the derivative F´(x) and then comparing gives $g(x)$ and $h(x)$ using equations 4 and 5 as

$$g(x) = \sum_0^6 \alpha_i \, x^i \text{ with } \alpha_5 = 0 \ , \ h(x) = \sum_0^8 \beta_j \, x^j \text{ with } \beta_1 = \beta_7 = 0 \text{ and constant} = \frac{1}{8} .$$

In terms of a, b and c we get using equations (5)

$$g(x) = x^6 + \frac{3}{2}(2a-c)\,x^4 - \frac{5}{3}bc\,x^3 + \frac{5(2a-c)^2 - 4(b^2c - a^2)}{8}\,x^2 - \frac{11}{10}bc\,(2a-c)\,x + (11/10)\,(2a-c)(b^2c - a^2) \ , \tag{10}$$

$$h(x) = x^8 + 2(2a-c)\,x^6 - \frac{8}{3}bc\,x^5 + \frac{5(2a-c)^2 - 4(b^2c - a^2)}{4}\,x^4 - \frac{44}{15}bc\,(2a-c)\,x^3 + (22/5)\,(2a-c)(b^2c - a^2)x^2$$
$$+\left[-\frac{3}{160}(2a-c)^2(b^2c-a^2) + \frac{1}{8}(b^2c - a^2)^2 + \frac{33}{80}b^2c^2(2a-c)\right] . \tag{11}$$

Software derivation algorithms can be tested by verifying that $F'(x) = f(x)$ using $g(x)$ and $h(x)$ of (10) and (11).

A **specific example** given in the Wikipedia article on "Risch algorithm" (or decision procedure) from 1976 is the elementary function

$$f(x) = \frac{x}{\sqrt{x^4 + 10x^2 - 96x - 71}},$$

with anti-derivative

$$F(x) = -\frac{1}{8}\ln\Big((x^6 + 15x^4 - 80x^3 + 27x^2 - 528x + 781)\sqrt{x^4 + 10x^2 - 96x - 71} \\ - (x^8 + 20x^6 - 128x^5 + 54x^4 - 1408x^3 + 3124x^2 + 10001)\Big) + C.$$

The anti-derivative *F(x)* follows directly from equations 9-11 above with a=11, b= 4 and c=12, where 2a-c =10, 2bc = 96 and $b^2c - a^2$ = 71. This integral was, as far as we know, first solved in software by the Axiom CAS system, version July 2009, see figure 1 below.

```
daly@spiff: ~
daly@spiff:~$ export AXIOM=/space/lambda1/mnt/ubuntu
daly@spiff:~$ export PATH=$AXIOM/bin:$PATH
daly@spiff:~$ axiom -nox
                        AXIOM Computer Algebra System
                         Version: Axiom (July 2009)
                Timestamp: Sunday August 9, 2009 at 15:26:54
-----------------------------------------------------------------------------
   Issue )copyright to view copyright notices.
   Issue )summary for a summary of useful system commands.
   Issue )quit to leave AXIOM and return to shell.
-----------------------------------------------------------------------------

   Re-reading compress.daase   Re-reading interp.daase
   Re-reading operation.daase
   Re-reading category.daase
   Re-reading browse.daase
(1) ->
(1) -> f:=x/sqrt(x^4+10*x^2-96*x-71)

                     x
   (1)  ----------------------
         +--------------------+
         | 4       2
        \|x  + 10x  - 96x - 71
                                                  Type: Expression Integer
(2) -> integrate(f,x)

   (2)
   -
        log
                                                      +--------------------+
             6      4      3      2                   | 4       2               8
           (x  + 15x  - 80x  + 27x  - 528x + 781)\|x  + 10x  - 96x - 71  - x
         +
                6       5      4        3        2
           - 20x  + 128x  - 54x  + 1408x  - 3124x  - 10001
      /
        8
                                          Type: Union(Expression Integer,...)
(3) ->
```

Fig. 1 Application by Timothy Daly (dated August 2009).

It would be interesting to see to what extent modern CAS software can manipulate the integrals with parameters like a, b and c. In the Wikipedia article on the Risch algorithm it is stated that no elementary anti-derivative is found when the constant term under the square root is - 72. We have checked the recursive equations 5 above and find no solution for β_2 when γ_0 = -72. However, we have not tried the more general form for *F(x)* given in equation 6, hence the integral is from this point of view either non-elementary or of form given in equation 6

$$F(x) = v_0(x) + \sum c_i \, ln[v_i(x)].$$

The situation is a somewhat puzzling and gives maybe "a glimpse of the difficulties symbolic integration in software may meet" even with rather simple functions.

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4. Some references

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